



#### Problem 4

## Catalytic majorization

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## Problem

With a Theorem by Nielsen [N], we have a completely explicit criterion to decide, when one pure bipartite state can be converted to another such state, using only local quantum operations and classical communication. Using Nielsen's criterion one can show [JP1] that the following strange situation can happen: state  $A$  cannot be converted to state  $B$ , but  $A \otimes C$  can be converted to  $B \otimes C$ , where  $C$  is a suitably chosen entangled state, the "catalyst".

The problem is to give a similarly efficient criterion to decide which pure bipartite states can be converted into each other using a catalyst.

## Background

Here is Nielsen's criterion, which is a surprisingly direct rendering of the intuition that a "more entangled" pure state has a "more mixed" restriction. Thus  $A$  can be converted to  $B$  if and only if the eigenvalue sequence of the restriction of  $A$  is more mixed than that of  $B$  in the sense of majorization of probability vectors [Maj]. We say that one probability vector  $p = (p_1, \dots, p_n)$  is *more mixed* than another,  $q = (q_1, \dots, q_n)$  in the sense of majorization, if one and hence all of the following equivalent statements hold:

- For all  $k$  :  $\sum_{i>k} p_i \leq \sum_{i>k} q_i$ , provided both  $p$  and  $q$  are first brought into decreasing order.
- there is a doubly stochastic matrix  $D$  (positive entries, sum of all rows and all columns = 1) such that  $p = Dq$ .
- For every convex function  $f : \mathbf{R} \rightarrow \mathbf{R}$  :  $\sum_i f(p_i) \leq \sum_i f(q_i)$

The above problem can be rephrased completely in this context of majorization of classical probability vectors, since tensoring pure bipartite states means again tensoring of probability vectors for the eigenvalues of the reduced density operators. Thus we would like to characterize the order relation "catalytic majorization":

For some  $r$ ,  $(p \otimes r)$  is more mixed than  $(q \otimes r)$  in the sense of majorization.

The above list of equivalent characterizations of majorization points to a way a characterization might look like: we might look for convex functions  $f$ , such that  $p \mapsto \sum_i f(p_i)$  is monotone with respect to catalytic majorization, and hope to characterize the relation by such a set. One class of functions  $f$  with this monotonicity property is  $f(t) = t^x$ , for  $x > 1$ , because the corresponding functionals on probability vectors are multiplicative with respect to tensor products.

There is some further literature on the use of majorization for the characterization of pure state entanglement [V1], [JP2], [VJN1], [N2] and on catalysis [EW1] that may be useful.

## Partial results

Daftuar and Klimesh have shown that the dimension of the required catalyst is in general unbounded [DK].

## References

- [DK] S. Daftuar and M. Klimesh, Phys. Rev. A **64**, 042314 (2001) and quant-ph/0104058 (2001).
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- [JP1] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 3566-3569 (1999) and quant-ph/9905071 (1999).
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- [N] M. A. Nielsen, Phys. Rev. Lett. **83**, 436-439 (1999) and quant-ph/9811053 (1998). The original proof in this paper can be simplified considerably.
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- [V1] G. Vidal, Phys. Rev. Lett. **83**, 1046-1049 (1999) and quant-ph/9902033 (1999).
- [VJN1] G. Vidal, D. Jonathan, and M. A. Nielsen, Phys. Rev. A **62**, 012304 (2000).
- [Maj] We have avoided the use of a comparison symbol, or the terminology “p is majorized by q”, because there are different conventions in the literature. There is a rich literature on the subject, starting with the still to be recommended classic  
G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge UP (1934).  
Further standard references are  
A. W. Marshal and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press (1979)  
and, in the quantum context,  
R. Bhatia, *Matrix Analysis*, Springer (1996).